

More on 't Hooft loops in $\mathcal{N} = 4$ SYM

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ABSTRACT: We study supersymmetric 't Hooft loop operators in $\mathcal{N} = 4$ super Yang-Mills, generalizing the well-known circular 1/2 BPS case and investigating their S-duality properties. We derive the BPS condition for a generic line operator describing pointlike monopoles and discuss its solutions in some particular case. In particular, we present the explicit construction of the magnetic counterpart of Zarembo and DGRT Wilson loops and provide the general dyonic configurations for an abelian gauge group. The quantum definition of these supersymmetric 't Hooft loop operators is carefully discussed and we attempt some computations to next-to-leading order in perturbation theory.

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1 Introduction

Four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory still represents a fascinating playground to study dynamical properties of quantum fields at nonperturbative level. The rich mathematical structure and the large amount of supersymmetry constrain the theory enough to make it amenable to exact treatment. As an example, particular classes of loop operators, that are some of the most important observables in $\mathcal{N} = 4$ SYM, have been exactly calculated in a series of nice papers [1, 2, 3] using localization techniques.

Loop operators can be classified according to whether the running charges are electric or magnetic and describe heavy probe particles that move along a closed path in spacetime. Both classes are usually defined in any gauge theory respectively as order and disorder parameters and are useful to characterize the phases of the theory.

The analysis of loop operators in supersymmetric theories has been a wide field of research over the last fifteen years, starting from the seminal papers on the $\mathcal{N}=4$ electric case [4, 5]. In maximally supersymmetric SYM theory these operators become extremely useful for two further reasons: firstly their quantum expectation values provide interesting tools to check the AdS/CFT correspondence and indeed, for particular examples, exactly interpolating functions between weak and strong coupling regimes have been constructed. Furthermore they are also important in testing the action of S -duality, an exact quantum symmetry under which $\mathcal{N}=4$ SYM is believed to be invariant. Since this duality exchange the role of the electric and the magnetic degrees of freedom it maps electric operators (Wilson loops) to their magnetic counterparts ('t Hooft loops) and relates their expectation values.

The Wilson loop operator is usually defined as

$$\langle W(C) \rangle = \frac{1}{\dim(R)} \text{Tr}_R \left[\mathcal{P} \exp \left(i \oint_C A_\mu dx^\mu \right) \right] \quad (1.1)$$

and essentially measures the response of the gauge field to an external quark-like source passing around a closed contour C^1 : in ordinary QCD loops in the fundamental representation are used to distinguish the different phases of the theory. In $\mathcal{N}=4$ SYM the Wilson loop has been instead defined in [4, 5] as

$$\langle W(C) \rangle = \frac{1}{\dim(R)} \text{Tr}_R \left[\mathcal{P} \exp \left(\oint_C (i A_\mu \dot{x}^\mu + \phi^A \theta^A(s) |\dot{x}|) ds \right) \right] \quad (1.2)$$

and quite naturally the operator obtained couples not only with the gauge field but also with the six scalars of the theory. A necessary condition to preserve locally some amount of supersymmetry is that the couplings $\theta^A(s)$ ($A=1\dots 6$) in (1.2) satisfy the constraint $\theta^A \theta^A = 1$ [6]. In order to have a global BPS object the following and more stringent condition has to be satisfied

$$(i \Gamma_\mu \dot{x}^\mu(s) + \Gamma^A \theta^A(s) |\dot{x}|) \epsilon(s) = 0 \quad (1.3)$$

namely eq. (1.3) must admit a non trivial solution for

$$\epsilon(s) = \epsilon_0 + x^\mu(s) \Gamma_\mu \epsilon_1, \quad (1.4)$$

that is a conformal Killing spinor on \mathbb{R}^4 with ϵ_0 and ϵ_1 two sixteen-component Majorana-Weyl constant spinors. This requirement yields constraints either on the loop ($\dot{x}_\mu(s)$) or on the scalar couplings ($\theta^A(s)$) or on both quantities.

The most famous example of Wilson operator that satisfies the previous condition is the 1/2 BPS circular Wilson loop: it was suspected for a long time that a matrix model computation should be capture all the information of this operator [7, 8]. The conjecture has

¹Above R denotes the representations of the gauge group, where the quark-like external source transforms.

been rigorously proven in [1], where it has been shown that the path-integral of theory defined on S^4 localizes on a finite dimensional space and reduces to a simple gaussian matrix model. The circular Wilson loop, due to its invariance properties, can be computed as an observable in this matrix model, leading to the expected result.

A simple idea to solve eq. (1.3) and obtain supersymmetric Wilson operators was proposed by Zarembo in [6]: this construction is based on the additional requirement that the position of the loop on the scalar S^5 , defined by the functions $\theta^A(s)$, follows the tangent vector to the contour C . In this case one can show that for an arbitrary shape of the contour the loop preserves 1/16 of the original Poincaré supersymmetry but if the curve lies in a lower dimensional subspace of \mathbb{R}^4 we observe an enhancement of the supersymmetry.

Another interesting proposal has been put forward by Drukker, Giombi, Ricci and Trancanelli (DGRT) in [9, 10, 11]. There, the authors considered an new class of Wilson loops of arbitrary shape defined on a space-time three sphere S^3 : for a generic curve these loops preserve two supercharges but they discussed special cases in which also 4 and 8 supercharges are conserved. Of particular interest are the loops restricted to S^2 because perturbative computations suggest the equivalence with analogous observables in the purely bosonic two dimensional Yang-Mills theory on the two-sphere: there are different indications that this conjecture holds [12, 13, 14, 15, 16, 17] even if a complete proof of the conjecture is still missing (see [12] for more details). The above families do not exhaust all the possible supersymmetric Wilson loops with only bosonic couplings. Recently a systematic classification of all possible solutions to the equation (1.3) has been obtained by Pestun and Dymarsky [18] and new kinds of BPS observables have been defined.

In gauge theories it is also possible to introduce a completely different class of non-local observables, the disorder operators suggested by 't Hooft and commonly referred to as 't Hooft loops [19]. They are defined by specifying the singularity that the field configurations have near a path C on which the operators are supported and can be thought as the duals of the Wilson loops. Physically they insert a probe point-like monopole whose world-line is the given loop C and can be used to study the different phases of the theory as well. As shown in [19], while the expectation value of a Wilson loop operator in the confined phase satisfied the area law, the VEV of the 't Hooft loop satisfies a perimeter law, and viceversa in the deconfined phase. The prescription to calculate the VEV of these magnetic objects is to perform the path integral in the presence of the singularity along the path C , in the simple abelian case being the one associated to a Dirac monopole

$$F_{0i} = 0 \quad F_{ij} = \frac{m}{2} \epsilon_{ijk} \frac{x_k}{r^3}, \quad (1.5)$$

where $i, j=1,2,3$ and $m \in \mathbb{Z}$ is the magnetic charge of the monopole. In the non-abelian case with arbitrary gauge group G one embeds the construction by defining an homomorphism $\rho : U(1) \rightarrow G$ and requiring that the operator has along the loop a singularity that is the image under ρ of the abelian one. More in detail, the configuration for a 't Hooft line operator is given by

$$F_{0i} = ig^2 \frac{\theta}{16\pi^2} B \frac{x_i}{r^3} \quad F_{ij} = \frac{B}{2} \epsilon_{ijk} \frac{x_k}{r^3} \quad (1.6)$$

where B belongs to the Cartan subalgebra of g , the Lie algebra associated to the group G , and where we have also taken into account that, in the presence of a non zero θ angle, an electric field is generated via the famous Witten effect [20]. Since the element B has to obey to the Dirac quantization condition

$$\exp(2\pi i B) = \mathbf{1}, \quad (1.7)$$

one can show [21] that it can be identified as a coweight vector (magnetic weight) and takes values in the coweight lattice Λ_{cw} of G^2 . In $\mathcal{N} = 4$ SYM these operators can be defined as well, but to obtain classical configurations that preserve a part of the supercharges we have to ensure the vanishing of the following supersymmetric variation

$$\delta_\epsilon \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon(s) - 2\phi^A \Gamma^A \epsilon_1 = 0 \quad (1.8)$$

for a non-trivial Killing spinor $\epsilon(s)$. Its solution implies that also the scalar fields have a singularity that goes like $1/r$ near the loop C : for example one can easily verify that the singular configurations associated to the supersymmetric version of the straight line operator (1.6) are given by

$$F_{0i} = ig^2 \frac{\theta}{16\pi^2} B \frac{x_i}{r^3} \quad F_{ij} = \frac{B}{2} \epsilon_{ijk} \frac{x_k}{r^3} \quad \phi^9 = g^2 \frac{B}{8\pi} |\tau| \frac{1}{r}, \quad (1.9)$$

where only one of the six scalars of the theory is turned on and

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad (1.10)$$

is the generalized coupling constant. A well known non-trivial BPS magnetic operator is the circular 't Hooft loops. It has been studied carefully first in [22], where the definition of the quantum operator and the prescription to calculate its expectation value have been given. More recently in [2] the calculation of its VEV has been performed exactly with a localization procedure on S^4 : compared to the circular Wilson Loop's computation there is a new and crucial contribution arising from the equator of S^4 where the loop is supported. Furthermore in [23] the OPE analysis of the circular loop and the correlation functions with an arbitrary chiral primary operator have been studied.

The knowledge of the exact expression for the VEV of both Wilson and 't Hooft operators opens the possibility to check the action of S -duality on these classes of observables. S -duality is a generalization of the electro-magnetic duality and $\mathcal{N}=4$ SYM theory is conjectured to be invariant under it [24, 25, 26]. More specifically $\mathcal{N}=4$ SYM with gauge group G and generalized coupling constant τ is believed to be equivalent to the $\mathcal{N}=4$ SYM theory with the dual gauge group ${}^L G$ [21] and coupling constant ${}^L \tau$

$${}^L \tau = -\frac{1}{n_g \tau} \quad (1.11)$$

with $n_g = 1, 2, 3$ depending on the choice of the gauge group. Since S -duality maps electric onto magnetic degrees of freedom, it establishes a natural isomorphism between operators.

²Or equivalently in the weight lattice Λ_w of the dual group ${}^L G$

Explicit checks of the conjectured have been made for the action of the duality on chiral primary operators [27, 28, 29], surface operators [30, 31] and domain walls [32, 33]. For what concerns loop operators a nice calculation has been done in [22] where the prediction of the duality has been shown to hold to the next to leading order in the coupling constant expansion. Further and more general tests have been presented in [2], taking advantage of the exact results obtained from localization.

Since different classes of electric observables preserving less supersymmetry are usually defined in $\mathcal{N} = 4$ SYM theory (*e.g.* Zarembo and DGRT Wilson Loops) it is interesting to understand the properties of their magnetic counterparts and how S -duality acts on them. In this paper we will try indeed to define and investigate new classes of 't Hooft operators preserving less supersymmetry than the circular and the straight line 't Hooft loops.

Furthermore in $\mathcal{N} = 4$ SYM theory there are also BPS mixed Wilson-'t Hooft loops operators which source both electric and magnetic [3, 34, 35, 36][37]: one can describe such operators requiring that the fields have a singularity near the loop as in (1.9) and inserting into the path-integral a factor

$$\langle W(C) \rangle = \frac{1}{\dim(R)} \text{Tr}_R \left[\mathcal{P} \exp \left(i \oint_C A_\mu dx^\mu \right) \right], \quad (1.12)$$

where R is an irreducible representation of G_B the stabilizer of B [35]. These mixed operators are thus labeled by a pair (B, R) with B a magnetic weight and R a irrep. of G_B . It would also be interesting to investigate these dyonic operators since they are a very rich laboratory on which one can study the properties of S -duality.

The plan of the paper is the following: in Section 2, as starting point, we define mixed BPS loop operators in an $\mathcal{N} = 4$ Maxwell theory. We will analyze the supersymmetric properties of the singular configurations, we will calculate exactly their expectation value and show how the abelian S -duality acts on them. Section 3 is devoted to the analysis of 't Hooft loops in $\mathcal{N}=4$ SYM and in particular we define the magnetic counterpart of the Zarembo Wilson loop [6] and the DGRT Wilson loop [11]. In principle, using the formulae derived in that section, given a contour C and the scalar couplings θ^A that define a supersymmetric Wilson loop in $\mathcal{N}=4$ SYM (see [18] for the detailed classification), we would be able to obtain the dual BPS 't Hooft operator. In Section 4 we calculate the expectation value for a particular example of magnetic operator up to one loop order, checking the action of S -duality on this non-trivial function of the coupling constant. The last section is dedicated to the conclusions and to present some open problems and directions for future investigations. There are four appendices : in the first one we summarize the notation, in the second one we discuss the cancellation at one-loop level of the fermion and boson excitation spectra of the non-zero modes around some 't Hooft backgrounds, in the third one some technical details about the integration over the adjoint orbit of B are shown and in the last one some explicit configurations associated to 1/4 BPS 't Hooft operators are given.

2 Wilson - 't Hooft loops in $\mathcal{N} = 4$ supersymmetric Maxwell theory

2.1 Singular Configurations

The starting point of our analysis is the $\mathcal{N} = 4$ supersymmetric Maxwell theory. It contains a free photon, four free Weyl fermions and six neutral scalars. In this theory we will show how to construct a wide family of supersymmetric Wilson - 't Hooft loops. Depending on the specific form of the singular configurations these operators will preserve a number of super-symmetries which ranges from sixteen to two. In order to construct these objects, we have first to solve the classical Maxwell equations in the presence of a dyonic charge moving along a closed non-selfintersecting curve C . In Feynman gauge, the solution for the gauge potential can be easily expressed in terms of a contour integral

$$A_\mu(y) = \frac{\lambda}{2\pi} \int_0^{2\pi} ds \frac{\dot{x}_\mu(s)}{(y - x(s))^2}. \quad (2.1)$$

In (2.1) the overall constant λ is given by

$$\lambda = g^2 (n + m\tau) = g^2 \left(n + m \frac{\theta}{2\pi} \right) + 4\pi m i \quad (2.2)$$

where n, m in \mathbb{Z} are respectively the electric and the magnetic charges of the dyon and τ is the complexified coupling constant (1.10); the functions $x_\mu(s)$ parameterize the closed circuit C . If one evaluates the *generalized* field strength

$$F_{\mu\nu} = \underbrace{i \operatorname{Re} [\partial_\mu A_\nu - \partial_\nu A_\mu]}_{\text{electric part}} + \underbrace{\operatorname{Im} \left[\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \right]}_{\text{magnetic part}} \quad (2.3)$$

of the gauge connection (2.1), one can immediately check that it correctly describes the electro-magnetic fields of a (n, m) dyon and it possesses the correct singularity when approaching the loop C . In fact, in a small neighborhood of a point $\bar{x} \in C$, the circuit in (2.1) can be approximated by the straight line $x_\mu(s) \simeq \bar{x}_\mu + \dot{x}_\mu^0 s$ and the behavior of (2.3) for small r reads

$$F_{0i} \sim i n g^2 \frac{r^i}{r^3} \quad F_{ij} \sim m \epsilon_{ijk} \frac{r^k}{r^3}. \quad (2.4)$$

Here $r = |y - x|_\perp$ is the distance of the point y from the straight line and the coordinate i, j, k are transverse to the straight line. In order to define a BPS loop operator we have also to turn on the scalar fields, by introducing a sort of R -symmetry current proportion to six-component vector $\theta^A(s)$. The solution of the equations of motion for the fields ϕ^A can be easily determined and they take the form

$$\phi(y)^A = \frac{|\lambda|}{2\pi} \int ds \frac{\theta^A(s)}{(y - x(s))^2}, \quad (2.5)$$

where $A = 1 \dots 6$ and the scalar couplings θ^A are taken to obey the standard constraint $\theta^A \theta^A = 1$. The next step is to determine under which conditions the field configurations (2.1) and (2.5) define a loop operator which preserves a certain amount of supersymmetries.

2.2 Supersymmetric properties

The $U(1)$ gauge connection (2.1) and the scalar fields (2.5) define a BPS operators if they annihilate the supersymmetry transformation of the fermion field Ψ , *i.e.*

$$\delta\Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon - 2\Gamma^A \phi^A \epsilon_1 = 0 \quad (2.6)$$

where $M, N = 0, \dots, 9$, and $A = 4, \dots, 9$ and we used the usual ten dimensional notation (see app. A for additional details on our conventions). The parameter $\epsilon = \epsilon_0 + x_\mu \Gamma^\mu \epsilon_1$ in (2.6) is a conformal Killing spinor in \mathbb{R}^4 since ϵ_0 and ϵ_1 are two constant Majorana-Weyl spinors of opposite chirality. The former generates the Poincaré sector while the latter is responsible for the conformal supersymmetries. The condition (2.6) can be translated into an algebraic local constraint, which contains only the circuit and the coupling θ^A .

In fact let us consider the supersymmetric variation of the gaugino Ψ for the classical configurations (2.1) and (2.5):

$$\delta\Psi = |\lambda| \oint ds \left(\frac{(y-x)_\mu}{(y-x)^4} [-2i\Gamma^{\mu\nu} \cos\varphi \dot{x}_\nu - \epsilon_{\mu\nu\rho\sigma} \Gamma^{\rho\sigma} \sin\varphi \dot{x}_\nu - 2\Gamma^{\mu A} \theta^A] \epsilon(y) - 2\frac{\Gamma^A \theta^A}{(y-x)^2} \epsilon_1 \right), \quad (2.7)$$

where $|\lambda|$ and φ are the modulus and the phase of the complex number λ and $\epsilon(s) \equiv \epsilon_0 + \Gamma^\nu x_\nu(s) \epsilon_1$. By adding and subtracting the same term to eq. (2.7), it can be rearranged as follows

$$\begin{aligned} \delta\Psi = & |\lambda| \oint ds \left(\frac{(y-x)_\mu \Gamma^\mu}{(y-x)^4} [i\Gamma^\nu \cos\varphi \dot{x}_\nu + \Gamma^{1234} \Gamma^\nu \sin\varphi \dot{x}_\nu + \Gamma^A \theta^A] (\epsilon_0 + \Gamma^\nu x_\nu \epsilon_1) - \right. \\ & \left. - 2\Gamma_\mu \frac{(y-x)_\mu}{(y-x)^4} [i\Gamma^\nu \dot{x}_\nu + \Gamma^A \theta^A] \Gamma^\rho (y-x)_\rho \epsilon_1 - 2\frac{\Gamma^A \theta^A}{(y-x)^2} \epsilon_1 \right) \end{aligned} \quad (2.8)$$

The second line in eq. (2.8) can be shown to vanish identically by exploiting a little bit of *Diracology* and a trivial integration by parts. In the first line we have used the identity

$$\epsilon_{\mu\nu\rho\sigma} \Gamma^{\rho\sigma} = -2\Gamma^{\mu\nu} \Gamma^{1234}, \quad (2.9)$$

in order to eliminate the Levi-Civita tensor from (2.8). Therefore our configuration is supersymmetric only when

$$\delta\Psi = \oint ds \left(\frac{(y-x)_\mu \Gamma^\mu}{(y-x)^4} [i\Gamma^\nu \cos\varphi \dot{x}_\nu + \Gamma^{1234} \Gamma^\nu \sin\varphi \dot{x}_\nu + \Gamma^A \theta^A] (\epsilon_0 + \Gamma^\nu x_\nu \epsilon_1) \right) = 0. \quad (2.10)$$

Eq. (2.10) is, in turn, zero *iff*³ the integrand vanishes

$$[i\Gamma^\mu \cos\varphi \dot{x}_\mu + \Gamma^{1234} \Gamma^\mu \sin\varphi \dot{x}_\mu + \Gamma^A \theta^A] (\epsilon_0 + \Gamma^\nu x_\nu \epsilon_1) = 0. \quad (2.11)$$

³In order to see that we can note that if eq. (2.10) is equal to zero also the following expression is vanishing

$$\Gamma^\alpha \partial_\alpha \oint ds \left(\frac{(y-x)_\mu \Gamma^\mu}{(y-x)^4} [i\Gamma^\nu \dot{x}_\nu + \Gamma^A \theta^A] (\epsilon_0 + \Gamma^\nu x_\nu \epsilon_1) \right) = 0$$

that is equal to

This is the advertised local constraint which determines the BPS nature of our loop operator only in terms of the scalar couplings $\theta^A(s)$ and the circuit $x_\mu(s)$. For a merely electric loop, *i.e.* $\varphi = 0$, eq. (2.11) takes the simplified form

$$\left[i\Gamma^\mu \dot{x}_\mu + \Gamma^A \theta^A \right] (\epsilon_0 + \Gamma^\nu x_\nu \epsilon_1) = 0, \quad (2.12)$$

which is the usual BPS condition for a Maldacena-Wilson operator in $\mathcal{N} = 4$. For $\varphi \neq 0$ we can define

$$\epsilon'(x) = e^{-i\Gamma^{1234} \frac{\varphi}{2}} \epsilon(x), \quad (2.13)$$

which is obtained through a four-dimensional chiral rotation of the original spinor. This auxiliary quantity again obeys (2.12), *i.e.* the classification of the dyonic abelian loop operators reduces to that of the ordinary BPS Wilson loops.

The general solution of (2.12) was obtained in [18]. There the key step was to recast (2.12) in a covariant ten dimensional language by introducing the vector $v^M = \left\{ \frac{dx^\mu}{ds}, \theta^I(s) \right\}$. One finds

$$v^M(x) \Gamma_M \epsilon'(x) = 0 \quad (2.14)$$

where $\Gamma_M = (\Gamma_\mu, \Gamma_I)$ denotes, as usual, the ten dimensional Dirac matrices. Then to solve the above linear system, one considers ϵ' as given and looks for the couplings v^M which obey (2.14). One can distinguish two different families of solutions depending on whether or not the vector $u^M = \epsilon \Gamma^M \epsilon$ identically vanishes.

When $u^M \neq 0$, there is a unique solution of eq. (2.14) and it is given by $v^M = \kappa u^M$ with κ a complex number [18]. The resulting loops are the orbits of the conformal transformations generated by $Q_{\epsilon'(x)}^2$. If we consider only closed loops, we obtain operators defined on (p, q) *Lissajous figures* where p and q are integer numbers. The quantum properties for these operator in the pure electric case were studied in [40].

If u^M vanishes identically on a submanifold $\Sigma_\epsilon \subseteq \mathbb{R}^4$, ϵ is a *pure spinor* on Σ_ϵ and consequently it induces an almost complex structure J_ϵ on this region [18]. The possible solutions v^M of eq. (2.14) in a point $x \in \Sigma_\epsilon$ are then provided by all the anti-holomorphic vectors with respect to J_ϵ [18]. This result can be used to associate a supersymmetric loop operator to each closed contour γ in Σ_ϵ . An explicit construction of the vector v^M , modulo equivalence under the action of the superconformal group, for the possible choice of Σ can be found in [18].

Thus we have shown that if we define a dyonic operator as in (2.1, 2.5) with the configurations supported on a path C and characterized by the scalar coupling $\theta^A(s)$ in such a way that $(C, \theta^A(s))$ describe a supersymmetric Wilson loop, the mixed electric-magnetic configurations obtained preserve the same amount of supersymmetry of the electric operators. Moreover the supercharges preserved by the two classes of observables are not the same but are related by a peculiar transformation that can be read from (2.13).

$$\oint ds \, \delta(y - x) \left[i\Gamma^\nu \dot{x}_\nu + \Gamma^A \theta^A \right] (\epsilon_0 + \Gamma^\nu x_\nu \epsilon_1) = 0$$

and since this equation must hold for every value of the variable y one can choose $y = x(s)$ and integrating over s one obtains exactly the equation (2.11).

What we have learnt is not surprising, rather it is what S -duality predicts. Indeed it has already been shown in the literature [25, 42] that while all bosonic symmetry generators are mapped trivially under the Montonen-Olive duality, the supersymmetry generators are multiplied by a φ -dependent phase:

$$\bar{Q}_{\dot{\alpha}} \rightarrow e^{i\frac{\varphi}{2}} \bar{Q}_{\dot{\alpha}} \quad Q_{\alpha} \rightarrow e^{-i\frac{\varphi}{2}} Q_{\alpha} \quad (2.15)$$

depending on the chirality of the spinor. The physical reason is that the supersymmetry algebra in presence of a central extension is modified as

$$\begin{aligned} \{Q_L, Q_L\} &\sim iq + g = |\lambda| (i \cos \varphi + \sin \varphi) \\ \{Q_R, Q_R\} &\sim iq - g = |\lambda| (i \cos \varphi - \sin \varphi) \end{aligned} \quad (2.16)$$

with q and g the electric and the magnetic charges of the configuration. It's thus natural that after an electric-magnetic transformation the supercharges acquire a chirality dependent phase equal to $e^{\pm i\varphi/2}$.

2.3 Expectation values of the operators

Above we have constructed new BPS dyonic configurations. The next step is the calculation of their expectation value and we can do that by firstly evaluating the classical action

$$S_{\mathcal{N}=4Max}^0 = \frac{1}{2g^2} \left(\int_{R^4} \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_{\mu} \phi^A D^{\mu} \phi^A \right) - i \frac{\theta}{32\pi^2} \left(\int_{R^4} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \quad (2.17)$$

on the field configurations (2.1, 2.5). It is easy to calculate separately the three different contributions in (2.17). The scalar part reads

$$\frac{1}{2g^2} \int d^4y D_{\mu} \phi^A D^{\mu} \phi^A = \frac{\lambda^2}{2g^2} S_{12}, \quad (2.18)$$

while the gauge contribution and the theta term are respectively given by

$$\begin{aligned} \frac{1}{4g^2} \int d^4y F_{\mu\nu} F^{\mu\nu} &= -\frac{1}{2} \left[g^2 \left(n + \frac{m\theta}{2\pi} \right)^2 - \frac{4\pi^2 m^2}{g^2} \right] G_{12} \\ -i \frac{\theta}{32\pi^2} \left(\int_{R^4} d^4y F_{\mu\nu} \tilde{F}^{\mu\nu} \right) &= \frac{g^2}{2} \left[\frac{nm}{\pi} + \frac{m^2 \theta^2}{4\pi} \right] G_{12} \end{aligned} \quad (2.19)$$

The function S_{12} and G_{12} are defined as the following *formal* contour integral⁴

$$S_{12} = \frac{1}{4\pi^2} \oint ds dt \frac{\theta^A(s) \theta^A(t)}{(x(s) - x(t))^2} \quad G_{12} = \frac{1}{4\pi^2} \oint ds dt \frac{\dot{x}(s) \dot{x}(t)}{(x(s) - x(t))^2}. \quad (2.20)$$

Collecting all the contributions we obtain that the on-shell action evaluated on the classical configuration can be written as

$$S_{\mathcal{N}=4Max}^0 = -\frac{g^2}{2} \left[n^2 - \left(\frac{m\theta}{2\pi} \right)^2 - \frac{4\pi^2 m^2}{g^4} \right] G_{12} + \frac{g^2}{2} \left[\left(n + \frac{m\theta}{2\pi} \right)^2 + \frac{4\pi^2 m^2}{g^4} \right] S_{12} \quad (2.21)$$

⁴Separately the function S_{12} and G_{12} are ultraviolet divergent.

However this is not the end of the story since, in order to calculate the VEV of the dyonic operator, we have also to insert in the path integral a Wilson loop of the form [34]

$$W(C) = \exp \left[-\frac{1}{2\pi} \oint \left(n \operatorname{Re}[A_\mu \dot{x}^\mu] - \frac{|\lambda|}{g^2} \phi^A \theta^A \right) ds \right] \quad (2.22)$$

and to evaluate also its value on the classical configurations. Since the insertion of the operator contributes as

$$g^2 \left[-n^2 - \frac{nm\theta}{2\pi} \right] G_{12} + g^2 \left[\left(n + \frac{m\theta}{2\pi} \right)^2 + \frac{4\pi^2 m^2}{g^4} \right] S_{12} \quad (2.23)$$

the final expression is given by

$$\langle W H \rangle = \exp \left[-\frac{\lambda^2}{2g^2} (G_{12} - S_{12}) \right]. \quad (2.24)$$

This result is exact because we are dealing with a Gaussian theory and finite for locally BPS operators on smooth contour since a nice cancellation occurs when the variables s and t coincide during the integration [43].

3 't Hooft Loops in $U(N)$ $\mathcal{N}=4$ SYM

3.1 Introduction

This section is devoted to extend the abelian construction of the loop operators presented in the previous chapter to the non-abelian $\mathcal{N}=4$ SYM theory. More precisely we will consider the case in which the gauge group is $G = {}^L G = U(N)$ leaving the extension to a generic group for a future investigation. In such theory we will define two large families of magnetic operators that are dual to Zarembo and DGRT Wilson loops [6][9].

In order to construct them, following the standard procedure presented in [34], we have to embed the abelian construction into the non-abelian group G defining an homomorphism

$$\rho : U(1) \rightarrow G. \quad (3.1)$$

The most general homomorphism ρ maps $e^{i\alpha} \in U(1)$ to the diagonal matrix

$$G = \exp(i\alpha B) = \operatorname{diag}(\exp(im_1\alpha), \exp(im_2\alpha), \dots, \exp(im_N\alpha)) \quad (3.2)$$

with the N -plet of integers ${}^L w = (m_1, m_2, \dots, m_N)$ that identifies a coweight vector (magnetic weight) of G . The magnetic weights of $U(N)$ are in one-to-one correspondence with the Young tableaux containing m_l boxes in the l -th row and identify an irreducible representation of the group. As example if we choose to define the 't Hooft observable in the fundamental, the k -symmetric or the k -antisymmetric representation of $U(N)$ the diagonal matrix B will be respectively of the form

$$B_F = \operatorname{diag} \left(\underbrace{1, 0, 0, \dots, 0}_N \right), B_{k\text{-sym}} = \operatorname{diag} \left(\underbrace{k, 0, 0, \dots, 0}_N \right), B_{k\text{-ant}} = \operatorname{diag} \left(\underbrace{1, 1, \dots, 1}_k, \underbrace{0, \dots, 0}_{N-k} \right). \quad (3.3)$$

The singular gauge configurations associated to the magnetic operator can thus be constructed as

$$A_\mu(y) = B \int ds \frac{\dot{x}_\mu(s)}{(y - x(s))^2} \quad (3.4)$$

and using the same embedding one can define the scalar configurations

$$\phi^A(y) = B \int ds \frac{\theta^A(s)}{(y - x(s))^2}. \quad (3.5)$$

At this point it's not difficult to perform the supersymmetric variation of the non-abelian configurations

$$\delta_\epsilon \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon(s) - 2\phi^A \Gamma^A \epsilon_1 = 0 \quad (3.6)$$

and note that, since the color structure factorizes out, we obtain the same condition on the path C and the scalar couplings $\theta^A(s)$ found in the abelian case. As consequence the magnetic configuration defined as in (3.4, 3.5) with the same $(C, \theta^A(s))$ of a BPS Wilson loop will preserve the same number of supersymmetries of the electric observable.

In the following subsections we will show explicitly the singular configurations associated to two classes of BPS 't Hooft operators and verify their supersymmetric properties. More in detail we will analyze the straight line and the circular operator and provide their respective generalizations, *i.e.* Zarembo and DGRT 't Hooft loops.

3.2 1/2 BPS 't Hooft line

Let us begin considering the simplest case of 't Hooft loop operator namely when the observable is defined on a straight line. Without loss of generality we can parameterize the circuit and the scalar couplings as follow

$$x_\mu = (0, 0, 0, s), \quad \theta^A(s) = \theta_0^A \quad (3.7)$$

with the variable s ($-\infty < s < +\infty$) that describes the circuit, $\mu = 1..4$, $A = 0, 5..9$ and θ_0^A a six-dimensional constant vector that satisfies $\theta_0^A \theta_0^A = 1$. It's easy to see from (3.4, 3.5) that the associated configurations are given by

$$A_4(y) = \frac{B \pi}{|y_\perp|} \quad \phi(y)^A = \frac{\pi B \theta_0^A}{|y_\perp|} \quad (3.8)$$

where $y_\perp^2 = y_1^2 + y_2^2 + y_3^2$ and the other components of A_μ ($\mu = 1..3$) are identically zero. In order to show that these configurations preserve a part of the supersymmetries we have to verify that the following variation

$$\delta \Psi = \frac{1}{2} \Gamma^{\mu\nu} F_{\mu\nu} (\epsilon_0 + x_\mu \Gamma^\mu \epsilon_1) + \Gamma^{\mu A} \partial_\mu \phi^A (\epsilon_0 + x_\mu \Gamma^\mu \epsilon_1) - 2\tilde{\Gamma}_A \phi^A \epsilon_1 \quad (3.9)$$

vanishes identically when it is evaluated on (3.8). After some manipulations one can shows that it reduces to one independent constraint for the spinors ϵ_0 and ϵ_1

$$(\Gamma^{123} + \Gamma^A \theta_0^A) \epsilon_0 = 0 \quad (\Gamma^{123} - \Gamma^A \theta_0^A) \epsilon_1 = 0. \quad (3.10)$$

Since the matrices $(\Gamma^{123} \pm \Gamma^A \theta_0^A)$ are nilpotent we can easily conclude that the operator preserves half of the super Poincaré plus half of the super-conformal charges of the theory. Now we want to investigate the relation between these supercharges and those preserved by the Wilson operator defined as

$$\langle W(C) \rangle = \frac{1}{\dim(R)} \text{Tr}_R \left[\mathcal{P} \exp \left(\oint_C (i A_\mu \dot{x}^\mu + \phi^A \theta^A(s) |\dot{x}|) ds \right) \right], \quad (3.11)$$

where the circuit C and the scalar couplings are the same as in (3.7). To do that we have to compare the equations (3.10) with the supersymmetric variation of (3.11) given by

$$(i\Gamma^4 + \Gamma^A \theta_0^A) \epsilon_0 = 0 \quad (i\Gamma^4 - \Gamma^A \theta_0^A) \epsilon_1 = 0. \quad (3.12)$$

It is not difficult to check that, as expected from S -duality consideration (see previous section), the spinors ϵ_0^m and ϵ_1^m satisfying eq.(3.10) are related to the solutions of eq.(3.12) ϵ_0^e and ϵ_1^e by the four dimensional chiral rotation

$$\epsilon_0^m = \exp \left(\frac{i}{2} [\Gamma^{1234}] \frac{\pi}{2} \right) \epsilon_0^e \quad \epsilon_1^m = \exp \left(-\frac{i}{2} [\Gamma^{1234}] \frac{\pi}{2} \right) \epsilon_1^e. \quad (3.13)$$

3.3 Zarembo 't Hooft loop

A simple idea to solve the BPS equation for the Wilson operator given by

$$(i\Gamma_\mu \dot{x}_\mu(s) + \Gamma^A \theta^A(s)) (\epsilon_0 + x_\nu(s) \Gamma^\nu \epsilon_1) = 0 \quad (3.14)$$

was proposed in [6] by Zarembo. For an arbitrary shape of the loop C the author chose the scalar couplings $\theta^A(s)$ in such way that the resulting operators preserve at least two supercharges. If

$$\theta^A(s) = M_\mu^A \dot{x}_\mu(s) \quad (3.15)$$

with M_μ^A a rectangular 4×6 matrix that satisfies $M_\mu^A M_\nu^A = \delta_{\mu\nu}$, he showed that, considering only the super-Poincaré transformation, all the dependence from the circuit is totally dropped out in the equation (3.14). Only the dimensionality of the subspace in which the curve lies is important to determine the number of supercharges. Indeed the equation (3.14) can be rewritten as

$$(\Gamma_\mu - i\Gamma^A M_\mu^A) \epsilon_0 = 0 \quad (3.16)$$

and one can see that, since μ ranges from one to four and since each equation halves the number of preserved supercharged, for a generic contour the operator is 1/16 BPS.

Following the Zarembo's idea we can extend the magnetic line operator (3.8) and construct the Zarembo 't Hooft loops. They are defined by specify the singular configurations associated to the operators as

$$A_\mu(y) = B \int ds \frac{\dot{x}_\mu}{(y - x(s))^2} \quad \phi(y)^A = B \int ds \frac{M_\mu^A \dot{x}_\mu}{(y - x(s))^2}. \quad (3.17)$$

If we consider only the super-Poincaré transformations and choose a generic curve inside \mathbb{R}_4 , the supersymmetric variation of the magnetic observables is given by four independent equations

$$(\epsilon_{\mu\nu\rho\sigma} \Gamma^{\nu\rho\sigma} - \Gamma^A M_\mu^A) \epsilon_0 = 0 \quad (3.18)$$

and as consequence these operators are at least 1/16 BPS. However, as in the electric case if the curves lies on a subspace of \mathbb{R}_4 an enhancement of the supersymmetry occurs. If we compare the variation of the Zarembo Wilson operator to the eq. (3.18) we can note again that the preserved supercharges are not the same but, as previously underlined, are related by the transformation (3.13). Finally in order to know how many super-conformal charges are preserved one has to give the explicit shape of the loop, however for a generic circuit no super-conformal transformation will be preserved neither for the electric nor for the magnetic observable.

In the appendix (D.1) we will present an explicit configuration for a 1/4 BPS magnetic loop of Zarembo-type, *i.e.* an operator supported on a cusp at angle α , and we will study its supersymmetric properties.

3.4 1/2 BPS Circular 't Hooft Loop

The circular 't Hooft loop operator is already well known in the literature since the work of Kapustin [34] and here we only review briefly its construction using our notation. Let us start choosing the couplings that identify the operator as follow

$$x_\mu(s) = (\cos(s), \sin(s), 0, 0), \quad \theta^A(s) = \theta_0^A \quad (3.19)$$

where s ranges from zero to 2π . The classical configurations associated to the observable and obtained from (3.4) and (3.5) read as

$$A_1(y) = 2\pi B \frac{y_2 \left(1 + y^2 - \sqrt{(1 + y^2)^2 - 4y_\parallel^2}\right)}{y_\parallel^2 \left(\sqrt{(1 + y^2)^2 - 4y_\parallel^2}\right)} \quad A_2(y) = 2\pi B \frac{y_1 \left(1 + y^2 - \sqrt{(1 + y^2)^2 - 4y_\parallel^2}\right)}{y_\parallel^2 \left(\sqrt{(1 + y^2)^2 - 4y_\parallel^2}\right)} \quad (3.20)$$

$$\phi^A(y) = 2\pi B \frac{\theta_0^A}{\left(\sqrt{(1 + y^2)^2 - 4y_\parallel^2}\right)} \quad (3.21)$$

where $y_\parallel^2 = y_1^2 + y_2^2$ and $y^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$. The configurations found can be obtained from those presented in [13, 23, 34] and usually defined in $AdS_2 \times S^2$ by a simple conformal transformation that maps \mathbb{R}^4 to $AdS_2 \times S_2$ (see appendix A of [22] for the explicit form of the mapping⁵). The supersymmetric variation of the operator can be written as

$$(\Gamma^{34} \epsilon_1 + \Gamma^A \theta_0^A \epsilon_0) = 0 \quad (\text{Circular 't Hooft loop}) \quad (3.22)$$

⁵In $AdS_2 \times S^2$ the subgroup of the conformal group preserved by the operator acts as isometries and the explicit form of the fields is simpler.

and compared to the variation of the ordinary electric operator

$$(i\Gamma^{12}\epsilon_1 - \Gamma^A\theta_0^A\epsilon_0) = 0 \quad (\text{Circular Wilson loop}). \quad (3.23)$$

Both the operators preserve exactly one half of the supersymmetries of the theory and more precisely some linear combinations of the super-Poincaré and super-conformal charges.

3.5 DGRT 't Hooft Loops

Following the construction of the Wilson loops on S^3 introduced by Drukker *et al.* [11] we can define new BPS 't Hooft operators generalizing the 1/2 BPS circular magnetic observable presented in the previous subsection. Just for simplicity we present only the case in which the loop lies on a two sphere but no restriction arises if we want to extend the definition to a general loop on S^3 .

In order to construct these magnetic operators one has to choose the circuit as

$$x_\mu(s) = (x_1(s), x_2(s), x_3(s), 0) \quad 0 < s < 2\pi \quad (3.24)$$

with $x_1^2 + x_2^2 + x_3^2 = 1$ and the scalar couplings as

$$\theta^A(s) = M_K^A \epsilon^{IJK} \dot{x}_I(s) x_J(s) \quad (3.25)$$

where M_K^A is a generic three by six matrix that satisfies $M_I^A M_J^A = \delta^{IJ}$. With this ansatz the singular configurations are given by

$$A_I(y) = B \int_0^{2\pi} ds \frac{\dot{x}_I}{(y - x(s))^2} \quad \phi(y)^A = B \int_0^{2\pi} ds \frac{M_K^A \epsilon^{IJK} \dot{x}_I(s) x_J(s)}{(y - x(s))^2}. \quad (3.26)$$

To obtain the explicit configuration and study its supersymmetric properties one has to know the path C on which the operator is supported. However it's not difficult to show that for a generic curve inside the two sphere the singular configurations (3.26) define an 1/8 BPS object.

In the appendix (D.2) and (D.3) two types of BPS DGRT magnetic operators will be presented in detail : the first is supported on a latitude at polar angle θ_0 and the second on a wedge, a loop made of two arcs of length π connected at an arbitrary angle δ , *i.e.* two longitudes on the two-sphere. Their supersymmetric properties will be carefully analyzed and compared to those of the dual electric observables .

4 't Hooft loops expectation value

4.1 Introduction

In this section we compute explicitly the expectation value of some BPS 't Hooft operators up to next-to-leading order in perturbation theory generalizing the calculation presented in [22] for the 1/2 BPS circular loop. In a future investigation it could be really intriguing to go beyond the perturbative analysis deriving an exact expression for the VEV of these

magnetic observables using localization techniques.

In the semiclassical approximation the VEV of the 't Hooft operators is simply given by

$$\langle H \rangle = \exp^{-S_0}, \quad (4.1)$$

namely by the contribution to the $\mathcal{N} = 4$ SYM action of the singular configurations associated to the operators

$$A_\mu^0 = B \int ds \int \frac{d^4 p}{(2\pi)^4} \frac{\dot{x}_\mu(s) e^{ip(y-x(s))}}{p^2} \quad \phi_0^A = B \int ds \int \frac{d^4 p}{(2\pi)^4} \frac{\theta^A(s) e^{ip(y-x(s))}}{p^2}, \quad (4.2)$$

where as usual B is a magnetic weight of $U(N)$, $x_\mu(s)$ is the circuit on which the operators are supported and $\theta^A(s)$ are the scalar couplings. Writing down the bosonic part of the $\mathcal{N} = 4$ super Yang-Mills action as

$$S_{\mathcal{N}=4} = \frac{1}{2g^2} \int_{R^4} d^4 x \operatorname{tr} (F_{\mu\nu} F^{\mu\nu} + 2D^\mu \phi^A D_\mu \phi^A + [\phi^A, \phi^B]^2), \quad (4.3)$$

it's straightforward to evaluate it on the background (4.2) and find that

$$S_0 = \frac{4 \operatorname{Tr} B^2 \pi^2}{g^2} (G_{12} + S_{12}) \quad (4.4)$$

where G_{12} and S_{12} are defined in (2.20)⁶. Since this value is divergent, in order to regularize it one has to introduced a counter-term defined on the hypersurface Σ , the boundary of a solid tubular neighborhood of the contour C [13, 22]. Evaluating this term on the classical configurations one obtains

$$S_{\text{boundary}} = -\frac{2\pi^2 \operatorname{Tr} B^2}{g^2} S_{12} \quad (4.5)$$

that summed to the on-shell action makes correctly the regularized action convergent⁷

$$\langle H \rangle = \exp^{-S_0 - S_{\text{boundary}}} = \exp \left[-\frac{4 \operatorname{Tr} B^2 \pi^2}{g^2} (G_{12} - S_{12}) \right]. \quad (4.6)$$

In order to go beyond and carry out the quantum computation we have to perform the path integral expanding the quantum fields around the singular configurations

$$A^\mu = A_0^\mu + \hat{A}^\mu \quad \phi^A = \phi_0^A + \hat{\phi}^A \quad (4.7)$$

where $(\hat{A}, \hat{\phi})$ are the non-singular quantum fluctuation on which we have to integrate over [22]. To quantize the theory in the background of A_0^μ, ϕ_0^A we have also to fix the gauge imposing for example that

⁶The coefficient $\operatorname{Tr} B^2$ is equal to 1 if the operator is in the fundamental representation, k if it is in the k -antisymmetric irrep, k^2 if it is in the k -symmetric irrep and finally $\sum_i^N m_i^2$ for a generic representation.

⁷As already noted in section two the combination $G_{12} - S_{12}$ is finite even if G_{12} and S_{12} are separately divergent.

$$D_0^\mu \hat{A}_\mu - i[\phi_0^A, \hat{\phi}^A] = 0 \quad (4.8)$$

and then add to the $\mathcal{N}=4$ SYM action the gauge fixing plus the ghost contributions that in a ten dimensional notation can be written as

$$S_{gf+g} = \frac{1}{g^2} \int d^4x \operatorname{tr} \left[D_0^M \hat{A}_M D_0^N \hat{A}_N - \bar{c} D_0^M D_M c \right]. \quad (4.9)$$

At this stage the operator is gauge-dependent and thus we have to introduce a procedure to restore the gauge invariance. Following the reference [22] the idea is to include in the path integral definition the integration over the adjoint G -orbit of the magnetic weight B

$$O(B) = [B^g = g B g^{-1}, g \in U(N)] \quad (4.10)$$

which is diffeomorphic to the coset space $U(N)/H$ with H the invariance group of the weight B . On $O(B)$ one defines a metric and fixing carefully its normalization (for more detail and convention see Appendix B) one can perform the integration over the adjoint orbit obtaining

$$\int ds_{O(B)}^2 = \left(\frac{4\pi^2}{g^2} (G_{12} - S_{12}) \right)^{\dim(U(N)/H)/2} \operatorname{Vol}(U(N)/H) \prod_{\alpha>0} \operatorname{Tr}[E_\alpha, B]^2 \quad (4.11)$$

where E_α are the ladder operators associated to roots α of the Lie algebra $su(N)$ and the product is over the positive roots that don't belong to the invariance subgroup of G . Up to one loop order the expectation value of the 't Hooft loop is thus given by

$$\begin{aligned} \langle H \rangle = \exp \left[-\frac{4 \operatorname{Tr} B^2 \pi^2}{g^2} (G_{12} - S_{12}) \right] & \frac{[\det(i\Gamma^M D_{0M})]^{1/4} \det_g(-D_0^2)}{[\det_b(-\delta^{MN} D_0^2 + 2iF_0^{MN})]^{1/2}} \times \\ & \left(\frac{4\pi}{g^2} (G_{12} - S_{12}) \right)^{\dim(U(N)/H)/2} \operatorname{Vol}(U(N)/H) \prod_{\alpha>0} \operatorname{Tr}[E_\alpha, B]^2 \end{aligned} \quad (4.12)$$

where the one-loop determinants, arising from the explicit integration over the quantum fluctuation, have to be calculated. In this paper we don't perform in detail this calculation guessing that due the supersymmetric properties of the background fields a cancelation between the bosonic and the fermionic contributions occurs. Indeed this is what happens for the 1/2 BPS circular loop as shown in [22] and confirmed in [13]. The calculation for a generic 't Hooft operators seems more intricate and thus, sketching the computation for the Zarembo 't Hooft operators in Appendix B, we leave the full analysis for a future investigation.

In principle this is not the end of the story since another non-perturbative effect due to the so called monopole bubbling has to be considered. More in detail the presence of smooth monopole configurations that surround the singular monopole can screen the charge of the 't Hooft operator. Since the regular 't Hooft-Polyakov monopoles are labeled by the coroots, the net charge obtained from the screening can be found through the action of the

lower operator associated to the roots on the magnetic weight B . As consequence if the 't Hooft loop operator is in a certain representation all the weights that belong to it will contribute to the VEV of the observable.

4.2 Expectation value of DGRT-'t Hooft loops on S^2

In this subsection we will compute explicitly the expectation value of the DGRT 't Hooft loops on S^2 . Since we expect that their VEV is a non trivial function of the coupling constant their analysis constitutes an intriguing playground to study the action of the S -duality.

More in detail we consider the operators supported on a curve C that lies on a two-dimensional sphere and parameterize by

$$x_\mu(s) = (x_1(s), x_2(s), x_3(s), 0) \quad 0 < s < 2\pi \quad (4.13)$$

with $x_1^2 + x_2^2 + x_3^2 = 1$ and with the scalar couplings defined as in (3.25)

$$\theta^A(s) = M_K^A \epsilon^{IJK} \dot{x}_I(s) x_J(s). \quad (4.14)$$

As noted in the previous section generically these observables are BPS and preserve four supersymmetries that are linear combinations of the super Poincaré and the super conformal charges. For the explicit calculation we take our operator in the fundamental representation identified by the magnetic weight

$$B = \text{diag} \left(\underbrace{1, 0, \dots, 0}_N, 0 \right) \quad (4.15)$$

in such a way that we also avoid the problem of the monopole bubbling phenomena⁸. The specified singular configuration breaks the $G = U(N)$ symmetry to $H = U(N-1) \times U(1)$, the coset space G/H is isomorphic to \mathbb{CP}^{N-1} and has dimension $2(N-1)$. Using that for a generic path on S^2

$$G_{12} - S_{12} = 2 \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}^2} \quad (4.16)$$

where \mathcal{A} is the total area of the sphere and $\mathcal{A}_1, \mathcal{A}_2$ are the areas determined by the loop, the expectation value of the operators⁹ at the leading order is given by

$$\langle H_F \rangle = e^{-\frac{8\pi^2}{g^2} \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}^2}} \left(\frac{16\pi^2}{g^2} \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}^2} \right)^{N-1} \frac{1}{(N-1)!} . \quad (4.17)$$

This formula is valid at the leading order in the strong coupling expansion and can be interestingly compared with the expectation value of the S -dual electric operator in order to understand how the S -duality acts on such class of observables. We recall that for the

⁸No screening of the 't Hooft operators charge can occur for the fundamental and for the k-antisymmetric representations of $U(N)$.

⁹We guess that the one-loop determinant factor is trivial as for the circular magnetic operators. A proof of this fact is under investigation.

DGRT Wilson loops [9, 10, 12, 14, 15] a conjecture relates their expectation value to the VEV of the ordinary Wilson loops in the zero-instanton sector of the pure bosonic two-dimensional Yang-Mills theory on S^2 . Since in that case the theory is completely solvable [44], an exact expression for their VEV can be derived [45][46][47]

$$\langle W_F \rangle = \frac{1}{N} L_{N-1}^1 \left(g_{2d}^2 \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}} \right) \exp \left(-\frac{g_{2d}^2}{2} \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}} \right) \quad (4.18)$$

where g_{2d}^2 is the two-dimensional coupling constant. More precisely the conjecture states that the VEV of the DGRT-Wilson loop in $\mathcal{N} = 4$ SYM is obtained from the previous formula through a redefinition of the two dimensional coupling constant

$$g_{2d}^2 = \frac{g^2}{\mathcal{A}}. \quad (4.19)$$

Now performing a strong coupling expansion of the result, since the Laguerre polynomial in this regime is replaced by its argument to the maximal power, one obtains that

$$\langle W_F \rangle \simeq \frac{1}{(N-1)!} \left(g^2 \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}^2} \right)^{N-1} \exp \left(-\frac{g^2}{2} \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}^2} \right). \quad (4.20)$$

Remarkably the previous expression is identical to the formula (4.17) for the expectation value of the 't Hooft operator after the usual S -dual transformations on the coupling constant

$$g^2 \rightarrow g'^2 = \frac{16\pi^2}{g^2}. \quad (4.21)$$

Indeed since S -duality is a generalization of the ordinary electric-magnetic duality, it maps Wilson operators in a theory with coupling constant g to 't Hooft operators, defined on the same circuit and with the same scalar couplings, in a theory with coupling constant g' ¹⁰.

5 Summary and discussions

In this paper we have defined a large family of BPS 't Hooft loop operators in $\mathcal{N} = 4$ SYM theory. Indeed, even if the S -duality predicts their existence, no magnetic operators preserving less supersymmetries than the line and the circular 't Hooft loops has been analyzed in the literature.

Starting from the $\mathcal{N} = 4$ Maxwell theory, in section two we have introduced the mixed BPS Wilson-'t Hooft operators : after their definition we focused on their supersymmetric properties deriving the BPS condition for a generic line operator and on the computation of their expectation value.

In the following step the generalization of these observables to the non-abelian $U(N)$ $\mathcal{N} = 4$ SYM theory has been discussed. More in detail the magnetic configurations associated to Zarembo and DGRT 't Hooft loops have been explicitly shown and their supersymmetric

¹⁰Since we are dealing with the $G \stackrel{L}{=} G = U(N)$ gauge group the representation of the group and its dual are in one-to-one correspondence

properties have been carefully studied. Interestingly we have found that the supercharges preserved by the BPS Wilson loops and by their magnetic counterparts are not the same but are related by a four-dimensional chiral transformation as predicted from S -duality. Furthermore the quantum definition of the operators is discussed and the expectation value for a specific class of DGRT-'t Hooft loops has been calculated up to next-to-leading order in perturbation theory.

More work has to be done to have a complete picture of these operators. The first interesting problem to investigate regards the analysis of the dyonic loop observables in $\mathcal{N}=4$ SYM. Their construction, presented in section two only for the Maxwell theory, in $\mathcal{N}=4$ SYM seems to be more intricate respect to the definition of the pure magnetic operators. Indeed in order to describe them one has not only to impose the boundary conditions for the fields as in the magnetic case but also to introduce in the path integral a Wilson operator in a certain irreducible representation R of the stabilizer subgroup of the magnetic weight B [34] and the analysis of the supersymmetric properties can not be straightforward derived from the abelian case.

To complete the next-to-leading order calculation of the VEV of these operators performed in section (4), the evaluation of the one-loop determinants around the singular configurations is necessary. Perform explicitly the computation by diagonalizing the fluctuations operators seems quite intricate and indeed in [2] the relevant determinants have been calculated using the Atiyah-Singer index theorem. It would be intriguing to extend this computation for at least some classes of magnetic operators defined in this paper.

Different directions can be investigate as extensions of the present work. One can start from the study of the correlator of 't Hooft loops whose singular configurations associated should be a simple sum of that generated by the two 't Hooft loops on the two different paths. Furthermore in order to analyze the Operator Product Expansion (OPE) of the 't Hooft operators and extract additional information about the action of the S -duality one could investigate also their correlator with some local objects as the chiral primary operators (see [23] for the correlators between 1/2 BPS magnetic observable and CPOs). Another interesting direction for a future investigation is given by the definition and the study of these magnetic and mixed operators in some theories like the $\mathcal{N}=2$ SYM where, differently from the $\mathcal{N}=4$ SYM, the action of the S -duality is highly non trivial [48][49][50]. The generalization of the construction to some three dimensional theory would be equally interesting in order to investigate the properties of the 3d mirror symmetry [51][52].

Finally we could go further in the analysis on the relation between the DGRT loops operators in $\mathcal{N}=4$ SYM and the two dimensional Yang-Mills theory. In [13] the authors conjectured that the VEV of the 1/2 BPS circular 't Hooft operators is equivalent to the value of the two dimensional Yang-Mills partition function on the two sphere in a non-zero instanton sector. The situation is not so clear if we instead of the maximal circle we choose a generic path on S^2 . Thus It could be intriguing first to prove rigourously the conjecture and then identifies, if they exist, the correspondent two-dimensional observables of the DGRT 't Hooft loops.

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A Conventions

The four dimensional $\mathcal{N} = 4$ SYM can be obtained from the dimensional reduction of the $\mathcal{N} = 1$ SYM in $d = 10$ and its action can be written using the notation of [1] as

$$S = \frac{1}{g^2} \int d^4x \frac{1}{2} F_{MN} F^{MN} - \Psi \gamma^M D_M \Psi \quad (\text{A.1})$$

where all the field (A_M, ψ) are in the adjoint representation of the gauge group G and M, N take values in $\{0..9\}$. The covariant derivative and the field strength are given respectively by $D_M = \partial_M + A_M$ and $F_{MN} = [D_M, D_N]$. The space-time indices running from 1 to 4 have been indicated by Greek letters $\mu, \nu, \rho, \sigma...$ while the six directions associated with the R-symmetry have been labeled by the letter $A, B...$ with values in $\{0, 5, 6, 7, 8, 9\}$.

The 32×32 gamma matrices γ^M satisfy the anti-commutation rules

$$\{\gamma^M, \gamma^N\} = 2\delta^{MN} \quad (\text{A.2})$$

and in the Weyl representation can be taken in the following form

$$\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}$$

where the explicitly expression of the 16×16 matrices $\Gamma^M = (\tilde{\Gamma}^M)^\dagger$ can be found in the Appendix A of [1]. In this representation the Dirac spinor ψ splits into two sixteen component spinors of opposite chirality $\psi = (\psi_+, \psi_-)$ (respect to the chiral matrix $\gamma^{11} = -i\gamma^1 \dots \gamma^9 \gamma^0$).

The $\mathcal{N} = 4$ SYM action is left invariant by the super-conformal transformations

$$\delta_\epsilon A_M(x) = \epsilon(x) \Gamma^M \psi \quad (\text{A.3})$$

$$\delta_\epsilon \psi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon(x) - 2\tilde{\Gamma}^A \phi^A \epsilon_1 \quad (\text{A.4})$$

where the spinor $\epsilon(x)$ is a conformal Killing spinor on \mathbb{R}^4

$$\epsilon(x) = \epsilon_0 + \epsilon_1 x_\mu \Gamma^\mu, \quad (\text{A.5})$$

with ϵ_0 and ϵ_1 two 16 component constant spinors that generate the usual Super Poincaré and Super Conformal symmetries respectively.

B One-loop determinant around the singular configurations

The computation of the one loop determinants for the 1/2 BPS 't Hooft operator has been performed in [22] by diagonalizing the quadratic fluctuation operator around the singular configurations and more recently in [2] where instead the Atiyah-Singer index theorem has been used to perform the calculation. In this appendix following the approach of [22] we try to extend the computation to the BPS magnetic operators introduced in this paper. The one-loop determinant factor, obtained integrating out the quantum fluctuations, can be written in a compact ten-dimensional notation as

$$\frac{[\det(i\Gamma^M D_{0M})]^{1/4} \det_g(-D_0^2)}{[\det_b(-\delta^{MN} D_0^2 + 2iF_0^{MN})]^{1/2}} \quad (\text{B.1})$$

where the covariant derivative D_{0M} is in the background of the classical fields A_μ^0, ϕ_A^0 . In the paper we guess that a cancelation between the fermionic and bosonic determinants, that make the expression (B.1) trivial, occurs not only for the magnetic line operator but for every BPS configurations. The physical reason can be explained as follow. If we suppose to have an eigenfunction of the scalar operator a supersymmetric transformation that leaves the background invariant should rotate it into an eigenfunction of the fermionic operator and thus relate their corresponding eigenvalue. Since an analogous relation holds also between the gluino and the gluon spectra, rewriting (B.1) as a product over the eigenvalues and taking carefully their multiplicity, one could in principle see that the cancelation around the BPS background is a general fact of the supersymmetric theories [53]. Let us sketch briefly the idea for the Zarembo 't Hooft loops leaving the complete proof and the analysis of more complex DGRT 't Hooft operators for a future investigation.

We start by supposing to know the eigenfunctions of the bosonic operators A_N^λ ¹¹ with eigenvalue λ^2

$$(-\delta^{MN} D^2 + 2iF^{MN}) A_N^\lambda = \lambda^2 A_M^\lambda. \quad (\text{B.2})$$

From them one can construct the functions Ψ_λ as

$$\Psi^\lambda = \Gamma^{AM} D_A^0 A_M^\lambda \epsilon_0 \quad (\text{B.3})$$

where the spinor ϵ_0 is a solution of the BPS equations for the Zarembo 't Hooft operator (see section 3.3), *i.e.*

$$\frac{1}{2} \Gamma^{AB} F_{AB}^0 \epsilon_0 = 0. \quad (\text{B.4})$$

It's not so difficult to see that Ψ_λ defined in such way are eigenfunctions of the fermionic operator with eigenvalue λ since

$$(i\Gamma^M D_M^0 (i\Gamma^N D_N^0 \Psi_\lambda)) = (i\Gamma^M D_M^0 (-i\lambda^2 \Gamma^N A_N^\lambda \epsilon_0)) = \lambda^2 \Gamma^{AM} D_A^0 A_M^\lambda \epsilon_0 = \lambda^2 \Psi^\lambda \quad (\text{B.5})$$

Furthermore choosing the autofunction of the scalar operator as

¹¹Here all the indices A, B, M, N run from 0 to 9

$$\varphi^\lambda = \epsilon_0 \Gamma^{AM} D_{0A} A_M \epsilon_0 \quad (\text{B.6})$$

we have that

$$-D_0^2 \varphi^\lambda = -\epsilon_0 D_0^2 (\Gamma^{AM} D_{0A} A_M) \epsilon_0 = \epsilon_0 \lambda^2 \Gamma^{AM} D_{0A} A_M \epsilon_0 = \lambda^2 \varphi^\lambda \quad (\text{B.7})$$

Now rewriting the determinants as a product over the eigenvalues and taking carefully their correct multiplicity it's simple to see from (B.5) and (B.7) that the computation simplifies drastically giving the trivial result.

This conclude the calculation for the Zarembo 't Hooft loop but unfortunately this construction can not be straightforward generalized to the DGRT 't Hooft loops or in general to BPS operators preserving some super-conformal charges. Probably the fact that conformal symmetries is explicitly broken by the gauge fixing term makes the computation more intricate respect to the one presented here.

C Integration over the Adjoint Orbits

In this appendix we review some elements that we have used in the integration over the adjoint orbit of the coweight B . Let us consider the Lie algebra \mathfrak{g} associated to the group G . A common notation is to indicate with H_i the generator of the Cartan subalgebra of \mathfrak{g} and with $E_{\pm\alpha}$ the ladder operators associated with the root α .

Now let $B = b_i H_i$ a magnetic weight (coweight) with value in the Cartan algebra \mathfrak{h} , its adjoint orbit O_B is defined as

$$O_B \equiv \{B_g = g B g^{-1}, g \text{ in } G\} \quad (\text{C.1})$$

and it is diffeomorphic to the coset space G/H where H is the invariance group of B . Following [22] one can construct the metric on O_B from the Maurer-Cartan one-form

$$g^{-1} dg = i \left(\sum_i d\xi^i H_i + \sum_\alpha d\xi^\alpha E_\alpha \right) \quad (\text{C.2})$$

and find that

$$ds_{O_B}^2 = 2\mathcal{N} \sum_{\alpha > 0} \alpha(B)^2 \text{Tr}(E_\alpha, E_{-\alpha}) |d\xi^\alpha|^2 \quad (\text{C.3})$$

where \mathcal{N} is a normalization factor fixed from the value of the on-shell action and the sum is done over the positive root elements that don't belong to the invariance group of B , namely all E_α that satisfy $[E_\alpha, B] = \alpha(B) E_\alpha \neq 0$. The integration over the orbit give thus

$$\int ds_{O_B}^2 = \left(\frac{\mathcal{N}}{\pi} \right)^{\dim(G/H)/2} \text{Vol}(G/H) \prod_{\alpha(B) \neq 0, \alpha > 0} \alpha(B)^2. \quad (\text{C.4})$$

In the paper we have used the formula for the volume of the compact group $U(N)$ [54]

$$\text{Vol}(U(N)) = \frac{(2\pi)^{N(N+1)/2}}{\prod_{n=1}^{N-1} n!} \quad (\text{C.5})$$

D 't Hooft loop configurations

D.1 Zarembo - 't Hooft loop operators : 1/4 BPS cusp

In the following we present an explicit configuration for 1/4 BPS Zarembo 't Hooft loop, *i.e.* a magnetic operator supported on a cusp at angle α . The circuit can be parameterized as

$$x_\mu(s) = \{s, 0, 0, 0\} \text{ for } -\infty < s < 0, \quad x_\mu(s) = \{-s \cos \alpha, -s \sin \alpha, 0, 0\} \text{ for } 0 < s < +\infty \quad (\text{D.1})$$

while the scalar coupling and the matrix M_μ^A can be chosen without loss of generality as

$$\theta^A(s) = M_\mu^A \dot{x}^\mu(s) \quad M_\mu^A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The classical configurations associated to the operator are given by

$$A_1(y) = B \frac{\pi(1 - \cos \alpha) - 2 \text{ArcTan} \left[\frac{y_1}{y_{\perp 1}^2} \right] (1 + \cos \alpha)}{2 y_{\perp 1}^2} \quad A_2(y) = -B \sin \alpha \frac{\pi - 2 \text{ArcTan} \left[\frac{y_2}{y_{\perp 2}^2} \right]}{2 y_{\perp 2}^2} \quad (\text{D.2})$$

$$\phi^9(y) = -B \frac{\pi(1 - \cos \alpha) - 2 \text{ArcTan} \left[\frac{y_1}{y_{\perp 1}^2} \right] (1 + \cos \alpha)}{2 y_{\perp 1}^2} \quad \phi^8(y) = B \sin \alpha \frac{\pi - 2 \text{ArcTan} \left[\frac{y_2}{y_{\perp 2}^2} \right]}{2 y_{\perp 2}^2} \quad (\text{D.3})$$

where $y_{\perp 1}^2 = y_2^2 + y_3^2 + y_4^2$ and $y_{\perp 2}^2 = y_1^2 + y_3^2 + y_4^2$. Since the supersymmetric variation of the operator is given by two independent relations for the spinor ϵ_0

$$(\Gamma^{234} + \Gamma^9) \epsilon_0 = 0 \quad (\Gamma^{134} - \Gamma^8) \epsilon_0 = 0 \quad (\text{D.4})$$

as expected the magnetic "cusp" operator is a 1/4 BPS object.

D.2 DGRT - 't Hooft loop operators : 1/4 BPS latitude

For the DGRT magnetic operator supported on a latitude at polar angle θ_0 we parameterize the circuit as

$$x_\mu(s) = (\sin \theta_0 \cos s, \sin \theta_0 \sin s, \cos \theta_0, 0) \quad (\text{D.5})$$

with $0 < s < 2\pi$ and choose the matrix M that defines the scalar couplings (3.25) of the form

$$M_K^A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

even if other choices of the matrix are allowed due the invariance of the theory under the super-conformal group. As in the electric case the new 't Hooft operator, differently from the maximal circle one, couples with three of the six scalars of the theory. Their classical configurations have these behaviors

$$\phi^7(y) = -2\pi B \frac{y_1 \cos(\theta_0) (1 + y^2 - 2y_3 \cos(\theta_0) - y_c)}{y_{\parallel}^2 y_c} \quad (D.6)$$

$$\phi^8(y) = 2\pi B \frac{y_2 \cos(\theta_0) (1 + y^2 - 2y_3 \cos(\theta_0) - y_c)}{y_{\parallel}^2 y_c} \quad \phi^9(y) = 2\pi B \frac{\sin(\theta_0)^2}{y_c} \quad (D.7)$$

while for the gauge fields we have that

$$A_1(y) = -2\pi B \frac{y_2 (1 + y^2 - 2y_3 \cos(\theta_0) - y_c)}{2 y_{\parallel}^2 y_c} \quad A_2(y) = 2\pi B \frac{y_1 (1 + y^2 - 2y_3 \cos(\theta_0) - y_c)}{y_{\parallel}^2 y_c} \quad (D.8)$$

where in order to make more compact the expressions we made use of the notation

$$y_c = \sqrt{(1 + y^2 - 2y_3 \cos(\theta_0))^2 - 4y_{\parallel}^2 \sin(\theta_0)^2}, \quad y^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad y_{\parallel}^2 = y_1^2 + y_2^2.$$

From these expressions one can immediately check that in the case in which the latitude chosen is the equator they reduces to the well known 1/2 BPS circular configurations shown in the previous subsection.

The supersymmetric variation of the operator reduces to two independent relations for the spinor ϵ_0 and ϵ_1 :

$$(\Gamma^{82} + \Gamma^{71})\epsilon_1 = 0 \quad (D.9)$$

$$\Gamma^9 \epsilon_0 = (-\Gamma^{34} - (\Gamma^{82} + \Gamma^{93}) \cos \theta_0) \epsilon_1 \quad (D.10)$$

Again these equations become exactly those that describe the 1/4 BPS circular Wilson loops [39] at polar angle θ_0

$$(\Gamma^{82} + \Gamma^{71})\epsilon_1 = 0 \quad (D.11)$$

$$\Gamma^9 \epsilon_0 = (i\Gamma^{12} - (\Gamma^{82} + \Gamma^{93}) \cos \theta_0) \epsilon_1 \quad (D.12)$$

after the usual chiral transformation on the spinors (3.13).

D.3 DGRT - 't Hooft loop operators : 1/4 BPS wedge

In this sub-appendix we present the singular configurations associated to the 1/4 BPS magnetic operator supported on a "wedge" namely a circuit consisting of two longitudes separated by an azimuthal angle δ on S^2 . The circuit is specified by

$$x_\mu(s) = (\sin(s), 0, \cos(s), 0) \quad 0 < s < \pi \quad (\text{D.13})$$

$$x_\mu(s) = (-\cos\delta \sin(s), -\sin\delta \sin(s), \cos(s), 0) \quad \pi < s < 2\pi \quad (\text{D.14})$$

and the scalar couplings are given exactly as in the previous example by

$$\theta^A(s) = \epsilon_{IJK} M_I^A \dot{x}_J(s) x_K(s) \quad (\text{D.15})$$

with the matrix M shown in (D.2). The configurations of the gauge and the scalar fields are given by

$$A_1 = \frac{B}{2} \frac{2y_3(1+y^2)(\pi + 2\text{ArcTan}(\frac{2y_1}{y_c}) - 2y_c(\pi y_3 + y_1 \text{Log}[1 + \frac{4y_3}{1+y^2-2y_3}]))}{(y_1^2 + y_3^2)y_c} - \frac{1}{4} \cos\delta \frac{2y_3(1+y^2)(\pi + 2\text{ArcTan}(\frac{2y_t}{y_s}) - 2y_s(\pi y_3 + y_t \text{Log}[1 + \frac{4y_3}{1+y^2-2y_3}]))}{(y_t^2 + y_3^2)y_s} \quad (\text{D.16})$$

$$A_2 = -\frac{B}{2} \sin\delta \frac{2y_3(1+y^2)(\pi + 2\text{ArcTan}(\frac{2y_t}{y_s}) - 2y_s(\pi y_3 + y_t \text{Log}[1 + \frac{4y_3}{1+y^2-2y_3}]))}{(y_t^2 + y_3^2)y_s} \quad (\text{D.17})$$

$$A_3 = \frac{B}{2} \frac{2y_3(1+y^2)(\pi + 2\text{ArcTan}(\frac{2y_1}{y_c}) - 2y_c(\pi y_3 + y_1 \text{Log}[1 + \frac{4y_3}{1+y^2-2y_3}]))}{(y_1^2 + y_3^2)y_c} - \frac{1}{4} \cos\delta \frac{2y_3(1+y^2)(\pi + 2\text{ArcTan}(\frac{2y_t}{y_s}) - 2y_s(\pi y_3 + y_t \text{Log}[1 + \frac{4y_3}{1+y^2-2y_3}]))}{(y_t^2 + y_3^2)y_s} \quad (\text{D.18})$$

$$A_4 = 0 \quad (\text{D.19})$$

$$\phi^8 = \frac{B}{2} \left(\frac{(\pi + 2\text{ArcTan}(\frac{2y_0}{y_c}))}{y_c} - \cos\delta \frac{(\pi + 2\text{ArcTan}(\frac{2y_t}{y_s}))}{y_s} \right) \quad (\text{D.20})$$

$$\phi^7 = \frac{B}{2} \left(\sin\delta \frac{(\pi + 2\text{ArcTan}(\frac{2y_t}{y_s}))}{y_s} \right) \quad \phi_i = 0 \quad \text{for } i = 4, 5, 6, 9 \quad (\text{D.21})$$

where we have used for convenience the following notation

$$y_c = \sqrt{1+y^2-4y_\parallel^2} \quad y_s = \sqrt{1+y^2-4y_3^2-4(y_1 \cos\delta + y_2 \sin\delta)^2}$$

$$y_t = (y_1 \cos\delta + y_2 \sin\delta) \quad y^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 \quad y_\parallel^2 = y_1^2 + y_3^2.$$

As a first simple check is immediate to see that for $\delta = \pi$ these configurations reduce to the maximal circle ones (see section 3.4). Calculating the supersymmetric variation of the configurations we obtain two independent equations

$$\Gamma^{24} \epsilon_1 + \Gamma^8 \epsilon_0 = 0 \quad (\text{D.22})$$

$$[(\Gamma^{24} \cos \delta - \Gamma^{14} \sin \delta)] \epsilon_1 + (\Gamma^8 \cos \delta - \Gamma^7 \sin \delta) \epsilon_0 = 0 \quad (\text{D.23})$$

and as consequence the operator is 1/4 BPS. One can compare in detail these supercharges with those preserved by the dual electric operator and that are given by the solutions of

$$i\Gamma^{13} \epsilon_1 + \Gamma^8 \epsilon_0 = 0 \quad (\text{D.24})$$

$$[i(\Gamma^{13} \cos \delta + \Gamma^{23} \sin \delta)] \epsilon_1 + (\Gamma^8 \cos \delta - \Gamma^7 \sin \delta) \epsilon_0 = 0 \quad (\text{D.25})$$

to verify again that the S -duality acts on them as expected.

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